INVARIANT SOLUTIONS OF VISCOPLASTICITY EQUATIONS AND SOLUTION OF THE PROBLEM OF HELICAL MOTION OF A BINGHAM FLUID BETWEEN COAXIAL CYLINDERS

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Besides numerical methods of solution, exact solutions are very important for any equation, especially those which describe the motion of real media. They make it possible to reveal the qualitative characteristics of the phenomena studied, to construct asymptotic solutions of specific problems, and in some cases to find the actual solutions. On the basis of methods of group analysis of differential equations we construct two classes of exact solutions and give a mechanical interpretation of them. The second class of exact solutions is used to construct a closed solution of problems of the helical motion of a Bingham fluid between two coaxial cylinders.

1. Group Properties of the Equations of Steady-State Isothermal Flow of a Viscoplastic Medium. The systems of equations describing the slow flow of a viscoplastic medium can be written as [1, 2]

$$\nabla_i \tau^{ki} = \nabla^k H; \tag{1.1}$$
$$\nabla_i v^i = 0,\ i = 1, 2, 3, \tag{1.2}$$

where v^i are the components of the velocity vector; $\tau^{ki} = 2(\eta + \tau_0/h)\, e^{ki}$; $e^{ki} = 1/2(\nabla^k v^i + \nabla^i v^k)$; $h = \left[2\sum_{k,i}(e^{ki})^2\right]^{1/2}$; τ^{ki} are the components of the stress tensor; e^{ki} are the components of the strain-rate tensor; H is the reduced pressure; η is the viscosity; and τ_0 is the maximum shear stress.

Using the procedure of [3, 4], we find the group of continuous transformations admitted by the system (1.1), (1.2), after eliminating the stress-tensor components. This group is generated by the operators

$$\begin{gathered} X_i = \partial_{x_i}, \quad Y_i = \partial_{v^i}, \quad i = 1, 2, 3, \quad S = \partial_H, \\ N = x_i\partial_{x_i} + v^i\partial_{v^i}, \quad T_1 = x_2\partial_{v^3} - x_3\partial_{v^2}, \\ T_2 = x_3\partial_{v^1} - x_1\partial_{v^3}, \quad T_3 = x_1\partial_{v^2} - x_2\partial_{v^1}, \\ Z_1 = x_2\partial_{x_3} - x_3\partial_{x_2} + v^2\partial_{v^3} - v^3\partial_{v^2}, \\ Z_2 = x_3\partial_{x_1} - x_1\partial_{x_3} + v^3\partial_{v^1} - v^1\partial_{v^3}, \\ Z_3 = x_1\partial_{x_2} - x_2\partial_{x_1} + v^1\partial_{v^2} - v^2\partial_{v^1}. \end{gathered} \tag{1.3}$$

The Lie algebra (1.3) is a subalgebra of a special form of Lie algebra, which is admitted by plasticity equations with the Mises yield criterion [5]. This makes it possible to build a solution immediately, using the tables in [5].

2. Invariant Solutions. We construct an invariant solution on the subgroup $X_3 + \alpha Y_3 + \beta T_3 + 2C_1 S$. After elementary transformations this solution takes on the form

$$\begin{gathered} v^1 = A\left(x_2^2 - x_1^2 - 2x_3^2\right) - 2Bx_1x_2 - Cx_1 + Dx_2x_3, \\ v^2 = -2Ax_1x_2 + B\left(x_1^2 - x_2^2 - 2x_3^2\right) - Cx_2 - Dx_1x_3, \\ v^3 = \psi(x_2, x_3) + 4Ax_1x_3 + 4Bx_2x_3 + 2Cx_3, \\ H = f(x_1, x_2) + 2C_1x_3,\ A, B, C, D, C_1 - \text{const}. \end{gathered} \tag{2.1}$$

Here the functions ψ, f are determined from the equations

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$$\frac{\partial}{\partial x_1}\left[\left(\eta + \frac{\tau_0}{h}\right)(Dx_2 + \psi_{,1})\right] + \frac{\partial}{\partial x_2}\left[\left(\eta + \frac{\tau_0}{h}\right)(-Dx_1 + \psi_{,2})\right] = 2C_1,$$
$$h^2 = [(4Ax_1 + 4Bx_2 + 2C)^2 + (Dx_2 + \psi_{,1})^2 + (-Dx_1 + \psi_{,2})^2],$$
$$f(x_1, x_2) = \tau^{11}.$$

This solution can be used to analyze the stress–strain state of a viscoplastic rod, which is deformed by loads applied to the ends and torsional and bending moments that are equivalent to the longitudinal force.

Suppose that $D = 0$ and $\psi = \text{const}$, whereupon the velocity field has the form

$$v^1 = A\,(x_2^2 - x_1^2 - 2x_3^2) - 2Bx_1x_2 - Cx_1,$$
$$v^2 = -2Ax_1x_2 + B\,(x_1^2 - x_2^2 - 2x_3^2) - Cx_2,$$
$$v^3 = Ax_1x_2 + Bx_2x_3 + Cx_3,$$

and the components of the deviator of the stress tensor are

$$\tau^{11} = \tau^{22} = -\eta\,(4Ax_1 + 4Bx_2 + 2C) - \tau_0\varkappa/\sqrt{2},$$
$$\tau^{33} = -2\tau^{11},\ H = \tau^{11} + 2C_1x_3 + \text{const}$$
$$(\varkappa = \text{sign}\,(4Ax_1 + 4Bx_2 + C)).$$

In the case under consideration the medium is everywhere in the deformed state, excluding the plane $2Ax_1 + 2Bx_2 + C = 0$, which does not deform. This plane, therefore, can be taken to be a rigid plate, which deforms the half-space $2Ax_1 + 2Bx_2 + C < 0$.

Suppose now that $A = B = D = 0$ in the solution (2.1). In this case the velocity field is

$$v^1 = -Cx_1,\ v^2 = -Cx_2,\ v^3 = 2Cx_3 + \psi(x_1, x_2),$$
$$H = \tau^{11} + C_1x_3 + \text{const}, \tag{2.2}$$

and ψ is determined from

$$\eta\Delta\psi + \tau_0\,\text{div}\frac{\nabla\psi}{\sqrt{1 + (\nabla\psi)^2}} = 2C_1, \tag{2.3}$$

where for convenience we assume that $C = 1/\sqrt{6}$.

Suppose that $\psi = \psi(x_1)$, whereupon the solution (2.2) generalizes the solution of Myasnikov [6] about the compression of the viscoplastic layer by rigid plates to the three-dimensional case. Since the pertinent investigation does not differ fundamentally from [6], we do not give it here.

Suppose that $\psi = \psi(r)$, where r and θ are the polar coordinates. The equation for the desired function has the form

$$Z_r' + \frac{1}{r}Z = 2C_1,\quad Z = \eta\psi' + \frac{\tau_0\psi'}{\sqrt{1 + (\psi')^2}},$$

from which we get an algebraic equation for determining ψ':

$$\psi'^4 - 2\psi'^3K + \psi'^2(K^2 + 1 - S^2) - 2\psi'K + K^2 = 0$$
$$(K = C_2/(\eta r) + C_1r/\eta,\ C_2 = \text{const},\ S = \tau_0/\eta). \tag{2.4}$$

Clearly, it is difficult to find the solution of Eq. (2.4) in explicit form. We assume, therefore, that $S \gg 1$ and we look for the solution of this equation, expanding it in negative degrees of S. Suppose that $\psi' = A + BS^{-1} + O(S^{-2})$. Substituting this into (2.4), we find the solution

$$A = \frac{C_1r^2 + C_2}{\sqrt{r^2 - (C_1r^2 + C_2)^2}},\quad B = \frac{A\,(1 + A^2)}{\sqrt{1 + A^2 + AK}},$$

which can be used to describe viscoplastic flow in a compressing tube.

We construct the solution on the two-parameter subgroup $\langle X_3 + \beta S, Z_3 + \alpha S\rangle$. This must be sought in the form ($r\theta z$ is the cylindrical coordinate system)

$$H = \alpha\theta + \beta z + p(r),\ v_r = v_r(r),\ v_\theta = v_\theta(r),\ v_z = v_z(r).$$

For simplicity we set $v_r = 0$ and as a result we have

$$\frac{\partial p}{\partial r} = -\frac{\tau_{\theta\theta}}{r},\quad \frac{\alpha}{r} = \frac{\partial \tau_{r\theta}}{\partial r} + 2\frac{\tau_{r\theta}}{r},\quad \beta = \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r},$$

from which it follows that

$$\tau_{r\theta} = \alpha + 2C_2 r^{-2},\ \tau_{rz} = \beta r + 2C_1 r^{-1},\ C_i - \text{const.} \tag{2.5}$$

From (2.5) we can easily find the expressions

$$\begin{gathered} v_z'(\eta + \tau_0/h) = -0{,}5\beta r + C_1 r^{-1}, \\ (v_\theta' - v_\theta/r)(\eta + \tau_0/h) = -0{,}5\alpha + C_2 r^{-2}\left(h^2 = v_z'^2 + (v_\theta' - v_\theta/r)^2\right). \end{gathered} \tag{2.6}$$

Denoting $\tau_{r\theta} = \lambda\tau_{rz}$, we find $\lambda v_z' = (v_\theta' - v_\theta/r)$. When we substitute this into (2.6) we obtain the quadrature for determinining v_z: $v_z' = -\frac{\tau_0}{\eta\sqrt{1+\lambda^2}} - \frac{1}{\eta}\left(\frac{C_1}{r} - \frac{1}{2}\beta r\right)$, and find v_θ from $v_\theta' - v_\theta/r = \lambda v_z'$, which is easily integrated.

We use the results to construct the solution of the problem of the helical motion of a Bingham fluid between two coaxial cylinders. This problem arises, e.g., in hydrodynamic studies on drilling boreholes [1, 7, 8].

3. Helical Motion of a Viscoplastic Medium between Two Coaxial Cylinders. We consider the following flow scheme. A viscoplastic fluid, described by the Shvedov–Bingham model, flows between two coaxial vertical cylinders under the effect of a pressure drop Δp. The inner tube rotates with a constant angular frequency ω.

We choose a cylindrical coordinate system as the reference system. We direct the z axis along the axis of the tube and r along its radius. We confine the investigation to laminar flow modes. In the given coordinate system the velocity componet on the r axis is zero, i.e., $v_r = 0$. Moreover, by virtue of axisymmetry the sought quantities do not depend on θ. We assume that the ratio of the radius of the outer cylinder r_3 to the tube length L is small, i.e., $\varepsilon = r_3/L \ll 1$. In particular, this condition is satisfied for boreholes being drilled. Using the methods of similarity theory and dimensionless analysis, we can show that the general system of equations describing the flow of a viscoplastic medium is described, to within $O(\varepsilon^2)$, as

$$K_\tau \frac{\partial V_\theta}{\partial t} = (1 + \mathrm{I})\left[\frac{\partial \bar\tau_{\theta r}}{\partial R} + \frac{2\bar\tau_{\theta r}}{R}\right]; \tag{3.1}$$

$$K_\tau \frac{\partial V_z}{\partial t} = (1 + \mathrm{I})\left[2 + \frac{\partial \bar\tau_{rz}}{\partial R} + \frac{\bar\tau_{rz}}{R}\right]; \tag{3.2}$$

$$\frac{\partial p}{\partial R} = -\frac{(r_0\omega)^2\rho}{\Delta H}\frac{V_\theta^2}{R}. \tag{3.3}$$

Here

$$V_\theta = v_\theta/r_0\omega;\ V_z = v_z/v_0;\ H = (p + \rho g z)/(p_0 - p_1 - \rho g L);$$

$$t = \frac{\tau}{t_0};\ \bar\tau_{rz} = \frac{\tau_{rz}}{T_{rz}};\ \bar\tau_{r\theta} = \frac{\tau_{r\theta}}{T_{r\theta}};\ \bar h = \frac{h}{h_0};\ d = r_3 - r_0;\ h_0^2 = \left(\frac{v_0}{d}\right)^2 + \left(\frac{\omega r_0}{d}\right)^2;\ \Delta H = p_0 - p_1 - \rho g L;$$

$$T_{rz} = \frac{\eta v_0}{d}(1 + \mathrm{I});\quad T_{r\theta} = \frac{r_0\omega\eta}{d}(1 + \mathrm{I});$$

p_0 and p_1 are the pressures at the channel inlet and exit; r_0 and r_3 are the radii of the inner and outer cylinders; ρ is the density of the medium; t_0 is the time scale; τ is the time; v_0 is the scale for the longitudinal velocity, determined from

$$WR_0 = K_v\sqrt{1-(1-K_v)^2\,\mathrm{La}^2}/(1-K_v), \tag{3.4}$$

where $R_0 = r_0/d$; $\mathrm{La} = \Delta Hd/(2\tau_0 L)$ is the Lagrange criterion; $W = \omega\eta/\tau_0$ is the dimensionless angular velocity; $K_V = v_0/\omega_0$; $\omega_0 = \Delta Hd^2/(2\eta L)$ is the characteristic flow velocity when the maximum shear stress $\tau_0 = 0$; g is the free fall acceleration; $I = d\tau_0/\eta\sqrt{v_0^2 + (\omega r_0)^2}$ is the Ilyushin criterion; and $K_\tau = d^2\rho/\eta t_0$ characterizes the time of reconstruction of the velocity profiles in the flow. If it is substantially shorter than the characteristic time of the process t_0, then $K \ll 1$ and can be assumed to be a steady-state process. In particular, this condition is satisfied in the flow of the boring solution during drilling.

We consider the situation when $K_\tau \ll 1$. Then, clearly, system (3.1)-(3.3) accords with the factor-system for the invariant solution $\langle X_3 + \beta S,\ Z_3 + \alpha S\rangle$, if we set $\alpha = 0$ and $\beta = -2$ in the latter. The invariant solution under consideration can thus be used to describe the steady-state flow of a viscoplastic fluid between two coaxial cylinders, when the inner one rotates with constant angular velocity.

In order to find the respective constants in the invariant solution we must set boundary conditions, whose form depends on the type of flow. We distinguish two forms of flow, which differ from each other by the existence of a rigid core, where the stress does not exceed τ_0.

First we consider flow in which a core exists. In this case the boundary condition can be written as

$$R = R_0,\ V_\theta = 1,\ V_z = 0; \tag{3.5}$$

$$R = R_3,\ V_\theta = 0,\ V_z = 0; \tag{3.6}$$

$$R = R_1,\ |\tau| = \tau_0 \Leftrightarrow \frac{\partial V_\theta}{\partial R} - \frac{V_\theta}{R} = \frac{\partial V_z}{\partial R} = 0; \tag{3.7}$$

$$R = R_2,\ |\tau| = \tau_0 \Leftrightarrow \frac{\partial V_\theta}{\partial R} - \frac{V_\theta}{R} = \frac{\partial V_z}{\partial R} = 0; \tag{3.8}$$

$$\left.\frac{V_\theta}{R_1}\right|_{R=R_1} = \left.\frac{V_\theta}{R_2}\right|_{R=R_2},\quad V_z|_{R=R_1} = V_z|_{R=R_2}; \tag{3.9}$$

$$(R_2^2 - R_1^2) = -R_1\bar{\tau}_{-rz} - R_2\bar{\tau}_{rz}; \tag{3.10}$$

$$R_1^2\bar{\tau}_{\theta r} = R_2^2\bar{\tau}_{\theta r} \tag{3.11}$$

($R_1 = r_1/d$, $R_2 = r_2/d$, and r_1 and r_2 are the radii of the core). Integrating the equations obtained in Sec. 2 for the velocities and finding the respective constants from the boundary condition, we have

$$V = -\frac{R^2 - R_0^2}{2} + R_1^2 m \ln\frac{R}{R_0} - \frac{R_1^2}{2} J_z\left[\left(\frac{R}{R_1}\right)^2,\ \left(\frac{R_0}{R_1}\right)^2\right],\ R_0 \leqslant R \leqslant R_1; \tag{3.12}$$

$$V = -\frac{R^2 - R_3^2}{2} + R_1^2 m \ln\frac{R}{R_3} - \frac{R_1^2}{2} J_z\left[\left(\frac{R}{R_1}\right)^2,\ \left(\frac{R_3}{R_1}\right)^2\right],\ R_2 \leqslant R \leqslant R_3; \tag{3.13}$$

$$V_\theta = \frac{R}{R_0} + \left\{\frac{1}{2}R_1^2\left(\frac{R}{R_0^2} - \frac{1}{R}\right) - \frac{R}{2}J_\theta\left[\left(\frac{R}{R_1}\right)^2,\ \left(\frac{R_0}{R_1}\right)^2\right]\right\}\frac{\cos\alpha}{WR_0},\quad R_0 \leqslant R \leqslant R_1; \tag{3.14}$$

$$V_\theta = \left\{\frac{1}{2}R_1^2\left(\frac{R}{R_3^2} - \frac{1}{R}\right) - \frac{R}{2}J_\theta\left[\left(\frac{R}{R_1}\right)^2,\ \left(\frac{R_3}{R_1}\right)^2\right]\right\}\frac{\cos\alpha}{WR_0},\ R_2 \leqslant R \leqslant R_3, \tag{3.15}$$

where

$$m = x + 1 - \sqrt{\frac{1 + x(1 + \mathrm{La}_1^2)}{\mathrm{La}_1^2}};\quad \mathrm{La}_1 = \mathrm{La}\,R_0;$$

$$J_z[u, v] = \int_v^u \frac{(m-\xi)\,d\xi}{\sqrt{\mathrm{La}_1^2(m-\xi)^2\xi + \cos^2\alpha}};$$

$$J_\theta[u, v] = \int_v \frac{d\xi}{\xi\sqrt{\mathrm{La}_1^2(m-\xi)^2\xi + \cos^2\alpha}};$$

$$V = V_z/(1 + \mathrm{I});\ \sin\alpha = (m-1)\mathrm{La}_1;\quad \cos\alpha = -\sqrt{1 - \sin^2\alpha},$$

and the dimensions of the core are found from two transcendental equations:

$$1+\frac{K_R-1}{y}-x+m\ln\frac{x}{K_R}+J_z\left(1,\ \frac{1}{y}\right)-J_z\left(x,\ \frac{K_R}{y}\right)=0; \tag{3.16}$$

$$W=\frac{\cos\alpha}{2}\left\{1-y\frac{K_R-1}{K_R}-\frac{1}{x}+J_\theta\left[1,\ \frac{1}{y}\right]-J_\theta\left[x,\ \frac{K_R}{y}\right]\right\}. \tag{3.17}$$

Here $K_R = (R_3/R_0)^2$; $x = (R_2/R_1)^2$; $y = (R_1/R_0)^2$.

At a given pressure drop La an increase in the rotation velocity W entails an increase in the stresses in the flow and, hence, the width of the core $\delta = R_2 - R_1$ decreases. A flow mode, when there is no rigid core, is thus theoretically possible.

In this case we can easily obtain

$$V=-\frac{R^2-R_0^2}{2}+R_0^2x_0\ln\frac{R}{R_0}-\frac{\left(\sqrt{K_R}-1\right)}{2\mathrm{La}}J_z^0\left[\left(\frac{R}{R_0}\right)^2,\ 1\right]; \tag{3.18}$$

$$V_\theta=\frac{R}{R_0}+\frac{\mathrm{La}\,y_0}{2W}\left(R-\frac{R_0^2}{R}\right)-\frac{y_0R}{2WR_0}J_\theta^0\left[\left(\frac{R}{R_0}\right)^2,\ 1\right],\quad R_0\leqslant R\leqslant R_3, \tag{3.19}$$

where $J_z^0[u,v]=\int\limits_v^u\frac{(x_0-\xi)\,d\xi}{\sqrt{(x_0-\xi)^2\xi+y_0^2}}$; $J_\theta^0[u,v]=\int\limits_v^u\frac{d\xi}{\xi\sqrt{(x_0-\xi)^2\xi+y_0^2}}$; x_0 and y_0 are unknown parameters determined by

$$\mathrm{La}=\frac{\left(\sqrt{K_R}-1\right)}{(1-K_R+x_0\ln K_R)}J_z^0[K_R,\ 1]; \tag{3.20}$$

$$W=0{,}5y_0\left\{J_\theta^0[K_R,\ 1]-\mathrm{La}\left(\sqrt{K_R}+1\right)/K_R\right\}. \tag{3.21}$$

We note that Eqs. (3.18)-(3.21) hold only when the inequality $(R/R_0)^2[x_0 - (R/R_0)^2]^2 + y_0^2 > 1/(\mathrm{La}\,R_0)^2$ is satisfied. This means that at all points of the flow the stress exceeds the maximum shear stress.

4. Effect of the Rotation of the Inner Cylinder on the Hydraulic Friction Factor. Using Eqs. (3.12) and (3.13), we can easily find an expression for the hydraulic friction factor of an annular tube for a structural flow mode:

$$\xi=\frac{64}{\mathrm{Re}}f(R_0,\ R_3,\ R_1,\ R_2,\ W),$$

$$f=(R_3+R_0)/(R_1^4\Omega),\quad \mathrm{Re}=\beta\bar{v}_z(2d)/\eta,$$

$$\Omega=\left(\frac{K_R}{y}-x\right)^2+\left(1-\frac{1}{y}\right)\left(\frac{1}{y}+1-2x\right)+2m\left(x\ln\frac{K_R}{x}-1+\frac{1-K_R}{y}+x\right)+$$
$$+2\left\{J_\Omega\left[\frac{K_R}{y},\ x\right]-J_\Omega\left[\frac{1}{y},\ 1\right]-J_z\left[1,\ \frac{1}{y}\right](x-1)\right\},$$

$$J_\Omega[u,v]=\int\limits_v^u\frac{(t-v)(m-t)\,dt}{\sqrt{\mathrm{La}_1^2(m-t)^2t+\cos^2\alpha}}.$$

In a similar fashion we obtain an expression for ξ, when there is no rigid core:

$$\xi=(64/\mathrm{Re})f(R_0,\ R_3,\ W),$$
$$f=(R_3+R_0)/(R_0^4\Omega),$$
$$\Omega=-(K_R-1)^2+2x_0(K_R\ln K_R-K_R+1)-$$
$$-\frac{2\left(\sqrt{K_R}-1\right)}{\mathrm{La}}\int\limits_1^{K_R}\frac{(K_R-t)(x_0-t)\,dt}{\sqrt{(x_0-t)^2t+y_0^2}}.$$

The results of calculations at $(R_3/R_0)^2$ with these formulas are shown in Fig. 1, where $\delta = R_2 - R_1$,

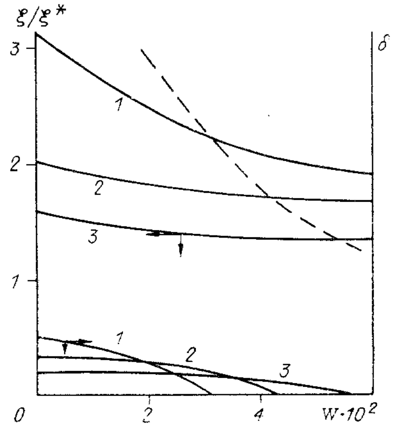

Fig. 1

$$\xi^* = \frac{64}{\mathrm{Re}} \frac{(R_3 - R_0)^2}{\left[R_3^2 + R_0^2 + (R_3^2 - R_0^2)/\ln(R_0/R_3)\right]},$$

ξ^* is the hydraulic friction factor of a Newtonian fluid in an annular channel lines 1-3 correspond to La = 2, 3, 5, and the dashed line corresponds to the boundary of the structural flow mode. We see that in the case of structural or laminar flow mode the hydraulic friction factor only decreases as the rotational velocity of the inner cylinder rises. This means that at a given flow rate, if a structural or laminar flow mode is realized, the voltage drop should decrease with increasing rotational velocity. We note that the higher the rotational velocity, the closer the hydraulic friction factor of a Bingham fluid is to the corresponding factor for a Newtonian fluid. We can conclude, therefore, that at higher rotational velocities the Bingham fluid behaves more like a Newtonian fluid. This is because the total stress in the flow increases, becoming substantially greater than the maximum shear stress, as the angular velocity of the inner cylinder increases.

This conclusion also makes it possible to explain the rise in the pressure with increasing rotational velocity, observed in [2, 8]. We must emphasize that these explanations must be sought from analysis of the turbulent flow of a Bingham fluid. Analyzing the dependence of ξ on Re and the Hendström criterion He = BiRe (Bi is the Bingham criterion), given in [8], we note that at $He < 5 \cdot 10^5$ the hydraulic friction factor is smaller than that of a Newtonian fluid and conversely at $He > 5 \cdot 10^5$. This means that if the indicated mechanism by which the properties of a Bingham and a Newtonian fluid approach each other persists in the turbulent mode, it becomes obvious that the pressure drop in a turbulent flow can increase or decrease as the rotational velocity of the inner cylinder grows. If the fluid is characterized by $He < 5 \cdot 10^5$, a growth of the rotational velocity increases the hydraulic friction factor and, hence, the pressure drop. If $He > 5 \cdot 10^5$, the reverse occurs: An increase in the rotational velocity causes the pressure drop to decrease.

It is also interesting to note the nontrivial behavior of the flow core. As the longitudinal pressure drop increases, as was to be expected, the thickness of the core decreases. In this case, however, the structural mode of flow persists at high velocities.

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REGARDING THE ARTICLE BY B. A. LUGOVTSOV "DETERMINATION OF THE MAIN FLOW PARAMETERS IN A SWIRL SPRAYER BY MEANS OF CONSERVATION LAWS" [1]

G. Yu. Stepanov

B. A. Lugovtsov examined the flow of an ideal incompressible fluid through a swirl sprayer with a cylindrical outlet. Figure 1 presents a sketch of the nozzle, with slight modifications from the unit depicted in Fig. 3 of [1]. Walls 1–1 and 2–2 are infinitely distant. The twisting of the flow is of a potential nature ($v_\varphi = \Gamma/(2\pi r)$, $\Gamma = \text{const}$), the twist parameter $A = R\Gamma/(2Q) = \text{const}$, and the free surface of the hollow core of the vortex is monotonic (without standing waves). The pressure $p_2 = \text{const} = 0$ in the meridional section.

We use the Bernoulli integral (incorrectly referred to in [1] as the energy integral) on the free surface at $z = -\infty$ and $z = \infty$

$$v_{1\varphi}^2/2 = v_{2\varphi}^2/2 + v_{2z}^2/2 = B = \text{const}$$

to obtain the discharge coefficient $\mu = Q/(\pi R^2\sqrt{2B})$ and $\overline{R}_1 \equiv R_1/R = A\mu$ as functions of A and $\overline{R}_2 = R_2/R$. The second of these functions is shown by the solid curves in Fig. 2. However, single-valued dependences of μ, $\overline{R}_1$, and $\overline{R}_2$ on A are seen in experiments. In Figs. 2 and 4, ∇ represents maxima on the curves.

G. N. Abramovich in 1943 and (independently) J. Taylor in 1948 proposed that a flow with a maximum discharge coefficient $\mu(\overline{R}_2)$ is realized for each specified parameter A [principle of maximum discharge (PMD)].* Here,

$$2\overline{R}_2^4 - A^2(1 - \overline{R}_2^2)^3 = 0, \quad \mu = (1 - \overline{R}_2^2)^{3/2}(1 + \overline{R}_2^2)^{-1/2}.$$

As is known from the hydraulic theory of spillways with a wide ramp and the linear problem of the fracture of a dam on a horizontal base, the PMD corresponds to the critical flow and follows from the continuity and Euler equations. It can be shown by analogy that if we assume that the thickness h of the layer of liquid in the nozzle outlet is small and the surface of the core of the vortex approximates the cylindrical surface of the outlet in the outlet section, the flow should be critical and have the Froude number

$$\mathrm{Fr} \equiv v_{2z} \Big/ \sqrt{h v_{2\varphi}^2/R} = 1 + O(\overline{h}^{3/2}), \quad \overline{h} \equiv h/R \equiv 1 - \overline{R}_2 = (2A)^{-2/3} + O(A^{-4/3}),$$

which for the specified value of A corresponds (to within quantities of the order of $\overline{h}^{5/2}$) to the maximum discharge coefficient μ.

For fairly large, realistic twist parameters ($A \geq 2$), use of the PMD in the hydraulic approximation has solid theoretical support and is backed by numerous experimental studies and is clearly the main technique employed in the design of centrifugal nozzles, various cyclone units, and other pieces of equipment whose operation involves swirling of the flow (for an example, see [2, Sec. 33; 3, pp. 90-94]).

*In Declaration No. 389 on 10.18.90, the State Commission on Inventions recognized G. N. Abramovich, L. A. Klyachko, I. I. Novikova, and V. I. Skobelkina as having discovered the "Law of fluid discharge in a swirled flow," in January of 1948.

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